Supersonic flow past a bluff body with a detached shock
Part II Axisymmetrical body

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SUMMARY

This paper considers the flow at high Mach number behind the curved shock formed when a supersonic stream impinges on an axisymmetrical body with a rounded nose.

The solution is obtained as a double expansion in
$$\delta = (\gamma-1)/(\gamma+1),$$
where γ is the adiabatic index, and M^{-2}; the expansion is developed to within terms of order $(\delta+M^{-2})^3$.

Expressions are obtained for the distance between the body and the shock, the radius of curvature of the shock compared with that of the body, and the pressure distribution on the body.

INTRODUCTION AND FUNDAMENTAL EQUATIONS

The technique evolved in Part I (Chester 1956) is used here to investigate supersonic flow past a blunt-nosed axisymmetrical body. The exposition is less detailed, for the ideas involved are essentially those of the previous paper.

The common axis of the body and of the shock wave formed ahead of it is in the direction of the uniform flow upstream of the shock. The x-coordinate is measured along this axis from an origin at the vertex of the shock. The radial coordinate is denoted by y.

Upstream of the shock the velocity, pressure and density are denoted by $(V,0)$, p_0, ρ_0 respectively. Corresponding quantities in the region of disturbed flow behind the shock are (Vu, Vv), $\rho_0 V^2 p$ and $\rho_0 \rho(\gamma+1)/(\gamma-1)$; as before, γ is the adiabatic index.

The equations of conservation of mass, momentum and entropy are, with $(\gamma-1)/(\gamma+1)=\delta$,

$$\left.\begin{aligned} \frac{\partial}{\partial x}(\rho u y)+\frac{\partial}{\partial y}(\rho v y) &= 0; \\ u\frac{\partial u}{\partial x}+v\frac{\partial u}{y\partial} &= -\frac{\delta}{\rho}\frac{\partial p}{\partial x}; \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{\delta}{\rho}\frac{\partial p}{\partial y}; \\ u\frac{\partial}{\partial x}(p\rho^{-\gamma})+v\frac{\partial}{\partial y}(p\rho^{-\gamma}) &= 0. \end{aligned}\right\} \qquad (1)$$

The first of these equations implies the existence of a stream function ψ such that $\delta\psi_y = \rho u y$, $\delta\psi_x = -\rho v y$. In the uniform flow ahead of the shock, $\psi = \frac{1}{2}y^2$ and is continuous across the shock. On the body, $\psi = 0$.

With the independent variables transformed to (ψ, y), equations (1) become

$$\left.\begin{aligned} y\frac{\partial p}{\partial\psi} - \frac{\partial u}{\partial y} &= 0; \\ u^2+v^2+(1+\delta)p/\rho &= 1+(1-\delta)\delta^{-1}M^{-2}; \\ \frac{\partial}{\partial\psi}\left(\frac{u}{v}\right)+\delta\frac{\partial}{\partial y}\left(\frac{1}{\rho v y}\right) &= 0; \\ p\rho^{-\gamma} &= f(\psi), \end{aligned}\right\} \tag{2}$$

where f is an arbitrary function of ψ.

We consider the particular case of a paraboloidal shock with unit radius of curvature at the nose. The equation of the shock profile is then $x = \frac{1}{2}y^2$, and the shock transition relations give the following boundary conditions to be satisfied on $\psi = \frac{1}{2}y^2$:

$$\left.\begin{aligned} u &= 1-(1-\delta)\left\{\frac{1}{1+y^2} - M^{-2}\right\}; \\ v &= (1-\delta)y\left\{\frac{1}{1+y^2} - M^{-2}\right\}; \\ p &= (1-\delta)\left\{\frac{1}{1+y^2} - \frac{\delta M^{-2}}{1+\delta}\right\}; \\ \rho &= \frac{1}{1+(1-\delta)\delta^{-1}M^{-2}(1+y^2)}. \end{aligned}\right\} \tag{3}$$

From these equations it follows that

$$f(\psi) = (1-\delta)\left\{\frac{1}{1+2\psi} - \frac{\delta M^{-2}}{1+\delta}\right\}\{1+(1-\delta)\delta^{-1}M^{-2}(1+2\psi)\}^{(1+\delta)/(1-\delta)}. \tag{4}$$

From the third of equations (2) we also have

$$\frac{u}{v} = \delta\frac{\partial}{\partial y}\int_{\psi}^{\frac{1}{2}y^2}\left(\frac{1}{\rho v y}\right)d\psi + \left(\frac{u}{v} - \frac{\delta}{\rho v}\right)_{\psi=\frac{1}{2}y^2}. \tag{5}$$

Equations (3) then give

$$\frac{u}{v} = y+\delta\frac{\partial}{\partial y}\int_{\psi}^{\frac{1}{2}y^2}\left(\frac{1}{\rho v y}\right)d\psi; \tag{6}$$

and, hence, the equation of the body is

$$x = \tfrac{1}{2}y^2+\delta\int_0^{\frac{1}{2}y^2}\left(\frac{1}{\rho v y}\right)d\psi. \tag{7}$$

As in Part I, it is not difficult to show that equation (6) includes automatically the stand-off distance between the body and the shock.

Basic solution

When $\delta = M^{-2} = 0$, equation (4) becomes

$$f(\psi) = \frac{1}{1+2\psi} + \delta^{-1}M^{-2},$$

and equations (2) are easily solved using the boundary conditions (3). The results are

$$\left.\begin{aligned} u = yv = \frac{(2\psi)^{1/2}y}{(1+2\psi)^{1/2}(1+y^2)^{1/2}}, \quad p = \frac{A}{2y(1+y^2)^{3/2}}, \\ \rho = \frac{(1+2\psi)A}{2y(1+y^2)^{3/2}\{1+\delta^{-1}M^{-2}(1+2\psi)\}}, \end{aligned}\right\} \tag{8}$$

where

$$A = [(2\psi)^{1/2}(1+2\psi)^{1/2} - \sinh^{-1}(2\psi)^{1/2} + y(1+y^2)^{1/2} + \sinh^{-1}y]. \tag{9}$$

The next step is to improve this approximation so as to obtain expressions for the velocity components which are uniformly valid as $\psi \to 0$. The expressions for u/v, p and ρ do not require modification; and so we have, correct to the *first* order of small quantities,

$$\begin{aligned} \frac{p}{\rho} &= f(\psi)\rho^{\gamma-1} = f(\psi)\{1+2\delta\log\rho\} \\ &= \frac{1}{1+2\psi}\left[1+\delta^{-1}M^{-2}(1+2\psi) - \delta - 2M^{-2}(1+2\psi) + \right. \\ &\qquad \left. + 2\{\delta + M^{-2}(1+2\psi)\}\log\left\{\frac{A(1+2\psi)}{2y(1+y^2)^{3/2}}\right\}\right]. \end{aligned} \tag{10}$$

This expression for p/ρ is substituted in the second of equations (2) to give

$$u^2+v^2 = \frac{1}{1+2\psi}\left[2\psi - 2(d+2\psi M^{-2})\log\left\{\frac{A(1+2\psi)}{2y(1+y^2)^{3/2}}\right\}\right], \tag{11}$$

where $d = \delta + M^{-2}$.

Equation (11) is now solved in conjunction with $u = vy$, and all terms which are *uniformly* $O(d)$ are discarded. The following relations are then obtained:

$$u = yv = \frac{y(2\psi+dB)^{1/2}}{(1+y^2)^{1/2}(1+2\psi)^{1/2}}, \tag{12}$$

where

$$B = 2\log\left\{\frac{2y(1+y^2)^{3/2}}{y(1+y^2)^{1/2}+\sinh^{-1}y}\right\}. \tag{13}$$

First-order approximation

From equations (8) and (12), we can now obtain the following expression for $\delta/\rho v$, which is uniformly correct as far as terms which are $O(d)$:

$$\frac{\delta}{\rho v} = \frac{2y(1+y^2)^2(d+2\psi M^{-2})}{A(1+2\psi)^{1/2}(2\psi+dB)^{1/2}}. \tag{14}$$

We can then obtain a first approximation to the equation of the body by the substitution of (14) in (7). The result is

$$x = \tfrac{1}{2}y^2 + 2d(1+y^2)^2 \int_0^{\frac{1}{2}y^2} \frac{d\psi}{(2\psi)^{1/2}(1+2\psi)^{1/2}A} +$$
$$+ M^{-2}(1+y^2)^2 \log\left\{\frac{2y(1+y^2)^{1/2}}{y(1+y^2)^{1/2}+\sinh^{-1}y}\right\} - \frac{2d^{3/2}B^{1/2}(1+y^2)^2}{y(1+y^2)^{1/2}+\sinh^{-1}y}. \quad (15)$$

Note that equation (15) contains terms which are $O(d)^{3/2}$, and is in fact correct to within $O(d^2)$.

To obtain the velocity components we must first calculate u/v from (6) which gives, correct to within $O(d^2)$,

$$u/v = y(1+g) + 2d\psi y^{-2}(1+y^2)^{1/2}(1+2\psi)^{1/2}(2\psi+dB)^{-1/2}, \quad (16)$$

where

$$g = -2d\psi y^{-2}(1+y^2)^{1/2}(1+2\psi)^{1/2}(2\psi+dB)^{-1/2} +$$
$$+ \frac{\partial}{\partial y}\left[2d(1+y^2)^2 \int_\psi^{\frac{1}{2}y^2} \frac{d\psi}{(2\psi)^{1/2}(1+2\psi)^{1/2}A} + \right.$$
$$+ M^{-2}(1+y^2)^2 \log\{2y(1+y^2)^{1/2}A^{-1}\} -$$
$$\left. - \frac{2d(1+y^2)^2}{y(1+y^2)^{1/2}+\sinh^{-1}y}\{(2\psi+dB)^{1/2}-(2\psi)^{1/2}\}\right]. \quad (17)$$

The form of the expression for u/v anticipates the fact that, near $y = 0$,

$$u \sim 2\delta\psi/\rho y^2 \sim 2d\psi/y^2$$

(the argument is similar to the corresponding one in Part I). Hence we may write

$$u = U + 2d\psi/y^2, \quad (18)$$

where U is $O(y^2)$ near $y = 0$. Equation (16) then gives

$$U/v = y(1+g); \quad (19)$$

and the singular term in (16) has now disappeared.

When these relations are substituted in (11) and the resulting equation solved for U, we find that

$$U = -\frac{2d\psi}{y^2} + y(1+2\psi)^{-1/2}(1+y^2)^{-1/2}\left(1+\frac{g}{1+y^2}\right)\times$$
$$\times\left[2\psi + 2(d+2\psi M^{-2})\log\left\{\frac{2y(1+y^2)^{3/2}}{A(1+2\psi)}\right\}\right]^{1/2}, \quad (20)$$

from which u and v are easily obtained using equations (14) and (15) respectively.

Finally the pressure is obtained by integration of the equation

$$\frac{\partial p}{\partial \psi} = \frac{1}{y}\frac{\partial u}{\partial y}$$

using the previously calculated expression for u and the boundary condition on $\psi = \frac{1}{2}y^2$ obtained from the last of equations (3). Although the variation of p with both ψ and y is required before the next approximation can be found, the complete expression is somewhat involved and we quote here only the expansion for small values of y and $\psi = 0$. The basic terms are given exactly, but the rest of the terms have errors which are $O(y^6)$:

$$p = (1+y^2)^{-3/2}\Big[\tfrac{1}{2}(1+y^2)^{1/2}+\frac{\sinh^{-1}y}{2y}+d\Big(-\frac{1}{2}-\frac{1201}{252}y^2+\frac{3277}{15120}y^4\Big)+ +M^{-2}\Big(1+\frac{7}{18}y^2+\frac{13741}{1080}y^4\Big)+\Big(\frac{8d}{3}\Big)^{3/2}\Big(\frac{4}{3}y^2-\frac{3}{5}y^4\Big)\Big]. \quad (21)$$

Second-order approximation

The precautions referred to in Part I also apply here; otherwise the process is straightforward, and only the final results are given. Apart from the basic terms, fourth and higher powers of y have been ignored.

The equation of the body is

$$x = \Big\{1-\Big(\frac{8d}{3}\Big)^{1/2}\Big\}d+\frac{26}{10}d^2-dM^{-2}+M^{-4}- -\frac{463}{168}d^2\Big(\frac{8d}{3}\Big)^{1/2}+\frac{13}{12}dM^{-2}\Big(\frac{8d}{3}\Big)^{1/2}-\frac{3}{2}M^{-4}\Big(\frac{8d}{3}\Big)^{1/2}+ +\tfrac{1}{2}y^2\Big[1+\frac{19}{6}d+\frac{2}{3}M^{-2}-\frac{47}{15}d\Big(\frac{8d}{3}\Big)^{1/2}+\frac{11173}{840}d^2-\frac{50}{21}dM^{-2}+\frac{23}{6}M^{-4}- -\frac{2593}{126}d^2\Big(\frac{8d}{3}\Big)^{1/2}+\frac{287}{20}dM^{-2}\Big(\frac{8d}{3}\Big)^{1/2}-\frac{47}{10}M^{-4}\Big(\frac{8d}{3}\Big)^{1/2}\Big]; \quad (22)$$

and the pressure on the body is given by

$$(1+y^2)^{3/2}p = \frac{1}{2}(1+y^2)^{1/2}+\frac{\sinh^{-1}y}{2y}-\frac{1}{2}d+M^{-2}+\frac{1}{8}d^2-\frac{3}{2}dM^{-2}+\frac{3}{2}M^{-4}- -y^2\Big\{\frac{1201}{252}d-\frac{7}{18}M^{-2}-\frac{32}{9}d\Big(\frac{8d}{3}\Big)^{1/2}+\frac{10009633}{55440}d^2+\frac{1}{252}dM^{-2}+ +\frac{725}{252}M^{-4}-\frac{3002}{189}d^2\Big(\frac{8d}{3}\Big)^{1/2}-\frac{40}{27}dM^{-2}\Big(\frac{8d}{3}\Big)^{1/2}-\frac{16}{3}M^{-4}\Big(\frac{8d}{3}\Big)^{1/2}\Big\}. \quad (23)$$

Discussion of results

Although, for reasons given in Part I, the analysis is carried out for the case of a shock having unit radius of curvature at the nose, the following results have been corrected to apply to a body having unit radius of curvature at the nose.

Figures 1 and 2 show respectively the radius of curvature of the shock and the distance between the body and the shock along the axis of symmetry for various values of d. The isolated point in each figure is an experimental result given by Oliver (1956). The curves are calculated from equation (22), except for the broken parts which are simply extrapolations through values of d for which the convergence of (22) may not be satisfactory.

A comparison of corresponding expressions in Part I and II shows that the convergence is poorer in the axisymmetrical case. Nevertheless, one or two general conclusions may be drawn. As in the two-dimensional problem, the stand-off distance seems to be virtually independent of M^{-2}, except in so far as this parameter influences d, and the single curve in figure 2 covers all cases.

As would be expected, under similar conditions both the radius of the shock and the stand-off distance are smaller in the axisymmetrical problem; a rough guide for the distance between the body and the shock is $2d$ in the two-dimensional case, and $\frac{3}{4}d$ in the axisymmetrical case.

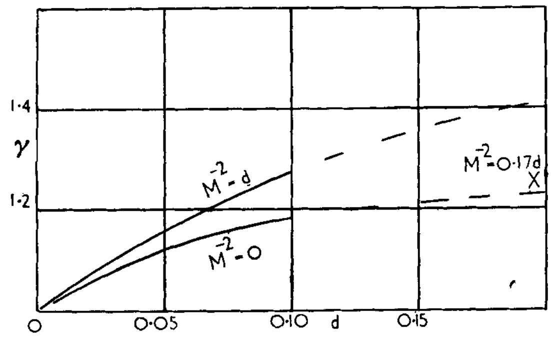

Figure 1. The radius of curvature of the shock on the axis of symmetry for various values of $d(=(\gamma-1)/(\gamma+1)+M^{-2})$. The unit of length is the radius of curvature of the body on the axis of symmetry. The cross denotes an experimental result.

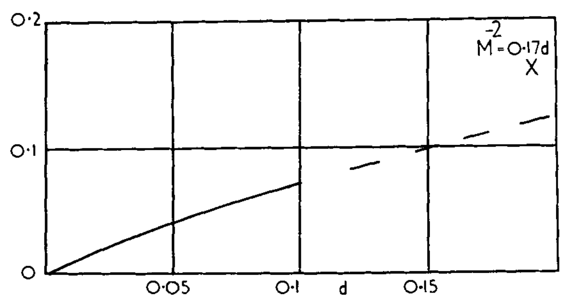

Figure 2. The distance between the body and the shock along the axis of symmetry for various values of $d(=(\gamma-1)/(\gamma+1)+M^{-2})$. The unit of length is the radius of curvature of the body on the axis of symmetry. The cross denotes an experimental result.

Because of the large factor multiplying d^2 (compared with the multiplier of d) in the coefficient of y^2 in equation (23), the expression for p cannot be regarded as reliable save for very small values of d in the neighbourhood of 0·01.

When $d = M^{-2} = 0$, the pressure distribution on the body is known exactly, and the ordinate of the sonic point is 0·680 compared with the value 0·629 in the two-dimensional case. According to calculations based on equation (23), the initial tendency is for the ordinate to decrease as d increases (and $M^{-2} = 0$) in contrast to an increase in the two-dimensional case (see Part I, figure 3), the ordinates agreeing in the two cases for a value of d around 0·02. But up to this value the decrease is only 0·3%, and the most that can be said is that initially there is no appreciable variation in the position of the sonic point.

Since the pressures on the body at the stagnation point and the sonic point are functions only of d and M^{-2} (see equations (29) and (30) of Part I), it follows that, with similar conditions, the pressures are the same at these corresponding points in the two-dimensional and axisymmetrical cases. Hence we may deduce that for sufficiently small values of d, the pressure distribution in the two cases do not differ appreciably in the neighbourhood of the nose.

References

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Oliver, R. E. 1956 *J. Aero. Sci.* **23**, 177.